QUANTUM FIELD THEORY

SECOND QUANTIZATION

Recap: quantum identical particles

$$|q_1, \dots, q_N\rangle_{\substack{B\\F}} = \frac{1}{\sqrt{N!}} \left\{ \begin{matrix} \text{sym} \\ \det \end{matrix} \right\} \begin{bmatrix} |q_1\rangle_1 & \cdots & |q_N\rangle_1 \\ \vdots & & \vdots \\ |q_1\rangle_N & & |q_N\rangle_N \end{bmatrix}$$

$\downarrow$

we'll use the notation

$$|q_1, \dots, q_N\rangle^{\pm} \quad \text{with } + = \text{Bosons}, \; - = \text{Fermions}$$

where $|q_i\rangle_j$

quantum # $\nearrow$ (on q_i) $\qquad$ $\nwarrow$ particle j (on j)

identical particles = we cannot say univocally which particle is in state q_i

$\Rightarrow$ need to consider all possibilities

Formulation inconvenient because

1. Scalar products $\rightarrow$ need to compute $(N!)^2$ terms
2. N° particles fixed $\rightarrow$ how do we create/destroy particles?
3. Large portions of Hilbert space do not contain physical states $\Rightarrow$ huge redundancy!

More convenient formalism:

SECOND QUANTIZATION

Central object → Fock Space

$$\mathcal{F} \equiv \bigoplus_{N=0}^{\infty} \mathcal{H}_N$$

$\mathcal{H}_N$ → Hilbert space of N particles

Obs: we introduce also $\mathcal{H}_0$ = Hilbert space with N=0 particles

→ Contains the Vacuum of the theory

In $\mathcal{F}$ we define a

CREATION OPERATOR

$$a_q^\dagger : \mathcal{H}_N \to \mathcal{H}_{N+1}$$

$$a_q^\dagger |q_1, \dots, q_N\rangle^{\pm} \equiv |q, q_1, \dots, q_N\rangle$$

$a_q^\dagger$ adds a particle with quantum # q at the beginning of the list $\{q_1, \dots, q_N\}$

In particular

$$|q_1, \dots, q_N\rangle^{\pm} = a_{q_1}^\dagger \cdots a_{q_N}^\dagger |0\rangle$$

unique vacuum
(= unique state of $\mathcal{H}_0$)

To guarantee that $|q_1, \dots, q_N\rangle^{\pm}$ = system of N bosons or fermions we need

$$|q_1, \dots, q, \dots, q', \dots, q_N\rangle^{\pm} = \pm |q_1, \dots, q', \dots, q, \dots, q_N\rangle^{\pm}$$

$$\Downarrow$$

we must have

$$a_q^{\dagger} a_{q'}^{\dagger} \mp a_{q'}^{\dagger} a_q^{\dagger} = 0$$

The adjoint of $a_q^\dagger$ (again in $\mathcal{F}$) is the

ANNIHILATION OPERATOR

How does a_q act on states?

1. Start with $N=2$ and require normalization

$$
{}^{\pm}\langle q_1' q_2' | q_1 q_2 \rangle^{\pm} = \frac{1}{\sqrt{2!}\sqrt{2!}} \left[\langle q_1' q_2' | \pm \langle q_2' q_1' | \right] \left[| q_1 q_2 \rangle \pm | q_2 q_1 \rangle \right]
$$

$$
= \delta(q_1' - q_1)\, \delta(q_2' - q_2) \pm \delta(q_1' - q_2)\, \delta(q_2' - q_1)
$$

Now use $\langle q_1' | a_q \equiv {}^{\pm}\langle q\, q_1' |$:

$$
\begin{aligned}
{}^{\pm}\langle q_1' | a_q | q_1 q_2 \rangle^{\pm} &= {}^{\pm}\langle q\, q_1' | q_1 q_2 \rangle \\
&= \delta(q-q_1)\,\delta(q_1'-q_2) \pm \delta(q-q_2)\,\delta(q_1'-q_1) \\
&= \delta(q-q_1)\,\langle q_1' | q_2 \rangle \pm \delta(q-q_2)\,\langle q_1' | q_1 \rangle \\
&= \langle q_1' | \underbrace{\Big[\delta(q-q_1)\,|q_2\rangle \pm \delta(q-q_2)\,|q_1\rangle \Big]}_{\text{must be } a_q |q_1 q_2\rangle^{\pm}}
\end{aligned}
$$

written in a more compact form

$$
a_q |q_1 q_2\rangle^{\pm} = \sum_{i=1}^{N} (\pm 1)^{i+1}\, \delta(q-q_i)\, |\bar{q}_i\rangle^{\pm}
$$

state without q_i

2. Generalization :

$$a_q \, | q_1, \dots, q_N \rangle^{\pm} = \sum_{i=1}^{N} (\pm 1)^{i+1} \, \delta(q - q_i) \, | q_1, \dots, \bar{q}_i, \dots, q_N \rangle^{\pm}$$

$$\Downarrow$$

a_q destroys a particle of quantum # q in all possible ways!

Special case : $a_q | 0 \rangle = 0$

Interesting information comes from the computation of $\quad a_{q'} a_q^\dagger \pm a_q^\dagger a_{q'}$

$$a_q \, a^\dagger_{q_0} |q_1, \dots, q_N\rangle^\pm = a_q \, |q_0, q_1, \dots, q_N\rangle^\pm$$

$=$ re-label $\{q_0, \dots, q_N\} \equiv \{\hat{q}_1, \dots, \hat{q}_{N+1}\}$ to make contact with previous notation

$$= \sum_{i=1}^{N+1} (\pm)^{i+1} \delta(q - \hat{q}_i) \, |\hat{q}_1, \dots, \overline{\hat{q}_i}, \dots, \hat{q}_N\rangle^\pm$$

$$= \delta(q - q_0) \, |q_1, \dots, q_N\rangle^\pm$$
$$+ \sum_{i=1}^{N} (\pm)^{i+2} \delta(q - q_i) \, |q, q_1, \dots \overline{q_i}, \dots, q_N\rangle^\pm$$

$$a^\dagger_{q_0} a_q |q_1, \dots, q_N\rangle^\pm = a^\dagger_{q_0} \sum_{i=1}^{N} (\pm)^{i+1} \delta(q - q_i) \, |\overline{q_i}\rangle^\pm$$

$$= \sum_{i=1}^{N} (\pm)^{i+1} \delta(q - q_i) \, |q_0, \overline{q_i}\rangle^\pm$$

To understand what is going on, let's compute the above expressions explicitly:

$$a_q a^{\dagger}_{q_0} |q_1, \dots, q_N\rangle^{\pm} = \delta(q-q_0)\, |q_1 \dots q_N\rangle^{\pm} \pm \delta(q-q_1) |q_0, \bar{q}_1\rangle^{\pm}$$
$$+ \delta(q-q_2) |q_0, \bar{q}_2\rangle^{\pm} \pm \delta(q-q_3) |q_0, \bar{q}_3\rangle^{\pm}$$
$$+ \dots$$

$$a^{\dagger}_{q_0} a_q |q_1, \dots, q_N\rangle^{\pm} = \delta(q-q_1) |q_0, \bar{q}_1\rangle^{\pm} \pm \delta(q-q_2) |q_0, \bar{q}_2\rangle^{\pm}$$
$$+ \delta(q-q_3) |q_0, \bar{q}_3\rangle^{\pm} + \dots$$

Thus signs inverted!

$$\left[a_q a^{\dagger}_{q_0} \mp a^{\dagger}_{q_0} a_q \right] |q_1 \dots q_N\rangle^{\pm} = \delta(q-q_0)\, |q_1, \dots, q_N\rangle^{\pm}$$

Morale:

BOSONS $\left[a_q, a^\dagger_{q'}\right] = \delta(q-q')$

FERMIONS $\left\{a_q, a^\dagger_{q'}\right\} = \delta(q-q')$

To complete the algebra of creation/annihilation operators, we can use

$$|q_1,\dots,q_N\rangle^{\pm} = a^\dagger_{q_1}\cdots a^\dagger_{q_N}|0\rangle$$

$$\oplus$$

$$|q_1,\dots q,\dots,q',\dots,q_N\rangle^{\pm} = \pm\, |q_1,\dots,q',\dots,q,\dots,q_N\rangle$$

$$\Downarrow$$

$$a^\dagger_q a^\dagger_{q'} \mp a^\dagger_{q'} a^\dagger_q = 0$$

h.c.

$$a_q a_{q'} \mp a_{q'} a_q = 0$$

Message : <u>the algebra is essentially the one of an harmonic oscillator</u>

Observables in Fock Space

Fundamental theorem:

ANY operator $\mathcal{O}$ can be expressed as a sum of products of creation/annihilation operators

$$\mathcal{O} = \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \int dq_1' \cdots dq_N' \, dq_1 \cdots dq_M \; C_{NM}(q, q') \, a^\dagger_{q_1'} \cdots a^\dagger_{q_N'} \, a_{q_1} \cdots a_{q_M}$$

No proof but we'll see some examples

1. Momentum

Idea: construct $\hat{\vec{P}}$ in terms of $a^\dagger_{\vec{p}}, a_{\vec{p}}$ in such a way that

$$\langle \vec{p}\,''|\,\hat{\vec{P}}\,|\vec{p}\,'\rangle = \vec{p}\,'\,\delta^3(\vec{p}\,'-\vec{p}\,'')$$

(we want the matrix elements to be the same, since matrix elements completely determine the operator)

Claim: $\hat{\vec{P}} \equiv \int d^3q\, \vec{q}\, a^\dagger_{\vec{q}}\, a_{\vec{q}}$

Check: $\langle \vec{p}\,''|\,\hat{\vec{P}}\,|\vec{p}\,'\rangle = \int d^3q\, \vec{q}\, \langle \vec{p}\,''|a^\dagger_{\vec{q}} a_{\vec{q}}|\vec{p}\,'\rangle$

$$= \int d^3q\, \vec{q}\, \delta^3(\vec{q}-\vec{p}\,')\,\delta^3(\vec{q}-\vec{p}\,'')\,\langle 0|0\rangle$$

$$= \vec{p}\,'\,\delta^3(\vec{p}\,'-\vec{p}\,'')$$

2. External potential in momentum representation

$U(\hat{\vec{X}})$ such that

$$\langle \vec{p}\,'' | U(\hat{\vec{X}}) | \vec{p}\,' \rangle = \int d^3x \, U(\vec{x}) \langle \vec{p}\,'' | \vec{x} \rangle \langle \vec{x} | \vec{p}\,' \rangle$$

$$= \int \frac{d^3x}{(2\pi\hbar)^3} e^{i(\vec{p}\,'-\vec{p}\,'')\cdot\vec{x}/\hbar} U(\vec{x})$$

$$\equiv \tilde{U}(\vec{p}\,' - \vec{p}\,'')$$

Claim:

$$U(\hat{\vec{X}}) = \int d^3q_1 \, d^3q_2 \, \tilde{U}(\vec{q}_1 - \vec{q}_2) \, a^\dagger_{q_1} a_{q_2}$$

Check: using $\langle \vec{p}\,'' | a^\dagger_{\vec{q}_1} a_{\vec{q}_2} | \vec{p}\,' \rangle$

$$= \delta^3(\vec{p}\,'' - \vec{q}_1) \, \delta^3(\vec{p}\,' - \vec{q}_2) \underbrace{\langle 0 | 0 \rangle}_{1}$$

we get

$$\langle \vec{p}''|U(\hat{\vec{x}})|\vec{p}'\rangle = \int d^3q_1\, d^3q_2\, \tilde{U}(\vec{q}_1-\vec{q}_2)\, \delta^3(\vec{p}''-\vec{q}_1)\,\delta^3(\vec{p}'-\vec{q}_2)$$

$$= \tilde{U}(\vec{p}''-\vec{p}')$$

3. Potential involving 2 particles

Consider $V(\hat{\vec{r}}_1, \hat{\vec{r}}_2) \rightarrow$ acts on the 2 particle sector of $\mathcal{F}$

$\Downarrow$

matrix elements must involve states of 2 particles

$$ {}^{\pm}\langle \vec{p}_1, \vec{p}_2 | V(\hat{\vec{r}}_1, \hat{\vec{r}}_2) | \vec{p}_3, \vec{p}_4 \rangle^{\pm} \equiv \left[V(\hat{\vec{r}}_1, \hat{\vec{r}}_2) \right]^{\pm}_{\vec{p}_1 \vec{p}_2 \vec{p}_3 \vec{p}_4} $$

Claim :

$$ V(\hat{\vec{r}}_1, \hat{\vec{r}}_2) = \int d^3 q_1 \cdots d^3 q_4 \left[V(\hat{\vec{r}}_1, \hat{\vec{r}}_2) \right]^{\pm}_{\vec{q}_1 \vec{q}_2 \vec{q}_3 \vec{q}_4} a^{\dagger}_{q_1} a^{\dagger}_{q_2} a_{q_3} a_{q_4} $$

Check : left as exercise

NUMBER OPERATOR

Claim: the operator

$$\hat{N} \equiv \int dq \; a_q^\dagger a_q$$

counts the # particles present in the state on which it is applied

Check:

a) $N=0$ $\qquad \hat{N}|0\rangle = \int dq \; a_q^\dagger \underbrace{a_q|0\rangle}_{0}$

$= 0$ <u>ok</u>

b) $N=1$ $\qquad \hat{N}|p\rangle = \int dq \; a_q^\dagger \underbrace{a_q|p\rangle}_{\delta(q-p)|0\rangle}$

$$= \int dq \; a_q^\dagger \, \delta(q-p) |0\rangle$$

$$= a_p^\dagger |0\rangle = |p\rangle$$

$|p\rangle$ = eigenstate of $\hat{N}$
with eigenvalue $= 1$

c) $N=2$

$$\hat{N} |p_1, p_2\rangle^\pm = \int dq \; a_q^\dagger a_q |p_1, p_2\rangle^\pm$$

$$= \int dq \; a_q^\dagger \left[\delta(q-p_1) |p_2\rangle \pm \delta(q-p_2) |p_1\rangle \right]$$

$$= |p_1 p_2\rangle^\pm \pm \underbrace{|p_2 p_1\rangle^\pm}_{\pm |p_1 p_2\rangle^\pm} = 2 |p_1 p_2\rangle^\pm$$

$\Rightarrow |p_1, p_2\rangle^{\pm}$ = eigenstate of $\hat{N}$ with eigenvalue $N=2$

$\Rightarrow$ we confirm that $\hat{N}$ counts the # of particles in the state it is applied to

Quantum Fields

We start using $\hat{a}_{\vec{p}}$, $\vec{p}$ = momentum

Hamiltonian : $\hat{H} = \frac{\hat{\vec{P}}^2}{2M} = \int d^3p \, \frac{\vec{p}^2}{2M} \, \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}}$

How do we pass to position space?

<u>Define</u> $\hat{a}^\dagger_{\vec{q}} \equiv \int d^3x \; u_{\vec{q}}(\vec{x}) \, \hat{\psi}^\dagger(\vec{x})$

Some weight function that depends on both $\vec{q}$ & $\vec{x}$

creation op. in position space

Also $\hat{a}_{\vec{q}} \equiv \int d^3x \; u^*_{\vec{q}}(\vec{x}) \, \hat{\psi}(\vec{x})$

Properties of $\hat{\psi}(\vec{x})$:

1. From $\quad a^\dagger_{\vec{q}}\, a^\dagger_{\vec{p}} \mp a^\dagger_{\vec{p}}\, a^\dagger_{\vec{q}} = 0$

$$\downarrow$$

$$0 = \int d^3x\, d^3y\; u_{\vec{q}}(\vec{x})\, u_{\vec{p}}(\vec{y}) \underbrace{\left(\hat{\psi}^\dagger(\vec{x})\,\hat{\psi}^\dagger(\vec{y}) \mp \hat{\psi}^\dagger(\vec{y})\,\hat{\psi}^\dagger(\vec{x})\right)}_{\text{must vanish}}$$

$$\boxed{\begin{aligned} &\hat{\psi}^\dagger(\vec{x})\,\hat{\psi}^\dagger(\vec{y}) \mp \hat{\psi}^\dagger(\vec{y})\,\hat{\psi}^\dagger(\vec{x}) = 0 \\ &\hat{\psi}(\vec{x})\,\hat{\psi}(\vec{y}) \mp \hat{\psi}(\vec{y})\,\hat{\psi}(\vec{x}) = 0 \end{aligned}}$$

2. from $\hat{a}_{\vec{q}}\hat{a}^{\dagger}_{\vec{p}} \mp \hat{a}^{\dagger}_{\vec{p}}\hat{a}_{\vec{q}} = \delta^3(\vec{p}-\vec{q})$

$$\downarrow$$

$$\delta^3(\vec{p}-\vec{q}) = \int d^3x\, d^3y\; u^*_{\vec{q}}(\vec{x})\, u_{\vec{p}}(\vec{y}) \left[\hat{\psi}(\vec{x})\hat{\psi}^{\dagger}(\vec{y}) \mp \hat{\psi}^{\dagger}(\vec{y})\hat{\psi}(\vec{x})\right]$$

can be true if

$$\hat{\psi}(\vec{x})\hat{\psi}^{\dagger}(\vec{y}) \mp \hat{\psi}^{\dagger}(\vec{y})\hat{\psi}(\vec{x}) = \delta^3(\vec{x}-\vec{y})$$

$$\oplus$$

$$\int d^3x\; u^*_{\vec{q}}(\vec{x})\, u_{\vec{p}}(\vec{x}) = \delta^3(\vec{p}-\vec{q}) \mapsto$$ Completeness condition

Message : $\hat{\psi}^{(\dagger)}(\vec{x})$ defined as above behave like normal creation/annihilation ops.

Since the $\{u_{\vec{q}}(\vec{x})\}$ system is complete (= basis), we can take it to be also orthonormal:

$$\int d^3p \; u_{\vec{p}}(\vec{x}) \, u^*_{\vec{p}}(\vec{y}) = \delta^3(\vec{x}-\vec{y})$$

But then we can invert

$$\hat{a}^\dagger_{\vec{p}} = \int d^3x \; u_{\vec{p}}(\vec{x}) \, \hat{\psi}^\dagger(\vec{x})$$

$\downarrow$ multiply by $u^*_{\vec{p}}(\vec{y})$ and integrate over d^3p

$$\begin{cases} \hat{\psi}^\dagger(\vec{y}) = \int d^3p \; u^*_{\vec{p}}(\vec{y}) \, \hat{a}^\dagger_{\vec{p}} \\ \hat{\psi}(\vec{y}) = \int d^3p \; u_{\vec{p}}(\vec{y}) \, \hat{a}_{\vec{p}} \end{cases}$$

What about Hamiltonian?

Claim: since $\langle \vec{x} | \hat{H} | \vec{y} \rangle = \frac{-\hbar^2 \vec{\nabla}_x^2}{2M} \delta^3(\vec{x}-\vec{y})$

then we must have

$$\hat{H} = \int d^3x \; \hat{\Psi}^\dagger(\vec{x}) \left[\frac{-\hbar^2 \vec{\nabla}^2 \hat{\Psi}(\vec{x})}{2M} \right]$$

Check: easy [write $\vec{\nabla}^2 \hat{\Psi}(\vec{x}) = \int d^3\ell \; \vec{\nabla}_x^2 \delta^3(\vec{\ell}-\vec{x}) \hat{\Psi}(\vec{\ell})$

to act with $\hat{\Psi}(\vec{\ell})$ on the right]

Connection between

$$\hat{H} = \int d^3x \; \hat{\psi}^\dagger(\vec{x}) \left[-\frac{\hbar^2 \vec{\nabla}^2}{2M} \hat{\psi}(\vec{x}) \right]$$

&

$$\hat{H} = \int d^3p \; \frac{\vec{p}^{\,2}}{2M} \; \hat{a}^\dagger_{\vec{p}} \, \hat{a}_{\vec{p}}$$

Start with

$$\hat{H} = \int d^3x \; \hat{\psi}^\dagger(\vec{x}) \left[-\frac{\hbar^2 \vec{\nabla}^2}{2M} \hat{\psi}(\vec{x}) \right] =$$

$$= \int d^3x \, d^3p \, d^3q \; u_{\vec{p}}(\vec{x}) \underbrace{\left[-\frac{\hbar^2 \vec{\nabla}^2}{2M} u^*_{\vec{q}}(\vec{x}) \right]} \hat{a}^\dagger_{\vec{p}} \, \hat{a}_{\vec{q}}$$

$\underline{\underline{\text{IF}}}$ $$-\frac{\hbar^2 \vec{\nabla}^2}{2M} u^*_{\vec{q}}(\vec{x}) = \frac{\vec{q}^{\,2}}{2M} u^*_{\vec{q}}(\vec{x})$$

$$= \int d^3p \, d^3q \underbrace{\left[\int d^3x \; u_{\vec{p}}(\vec{x}) \, u^*_{\vec{q}}(\vec{x}) \right]}_{\delta^3(\vec{p}-\vec{q})} \frac{\vec{q}^{\,2}}{2M} \; \hat{a}^\dagger_{\vec{p}} \, \hat{a}_{\vec{q}}$$

$$\frac{1}{\ } \int d^3p \, \frac{\vec{p}^{\,2}}{2M} \, \hat{a}^{\dagger}_{\vec{p}} \, \hat{a}_{\vec{p}}$$

$\Longrightarrow$ for the two approaches to be consistent, we need

$$u_{\vec{p}}(\vec{x}) = \underline{\text{Solution wave equation}}$$
$$\underline{\text{equation in position}}$$
$$\underline{\text{representation}}$$

In this case
$$u_{\vec{p}}(\vec{x}) = \frac{e^{i\vec{p}\cdot\vec{x}/\hbar}}{(2\pi\hbar)^{3/2}}$$

Heisenberg picture

Until now → operators are all time independent

To consider time dependency → Heisenberg picture

$$i\hbar\, \dot{\hat{a}}_{\vec{p}}(t) = \left[\hat{a}_{\vec{p}}(t), \hat{H}\right]$$

$$= \int d^3q \frac{\vec{q}^2}{2M}\left[\hat{a}_{\vec{p}}(t), \hat{a}^\dagger_{\vec{q}}(t)\,\hat{a}_{\vec{q}}(t)\right]$$

$$= \int d^3q \frac{\vec{q}^2}{2M}\left[\hat{a}_{\vec{p}}(t)\,\hat{a}^\dagger_{\vec{q}}(t)\,\hat{a}_{\vec{q}}(t) - \hat{a}^\dagger_{\vec{q}}(t)\,\hat{a}_{\vec{q}}(t)\,\hat{a}_{\vec{p}}(t)\right]$$

Commute

$$= \int d^3q \frac{\vec{q}^2}{2M}\left[\hat{a}_{\vec{p}}\,\hat{a}^\dagger_{\vec{q}}\,\hat{a}_{\vec{q}} - \hat{a}^\dagger_{\vec{q}}\,\hat{a}_{\vec{p}}\,\hat{a}_{\vec{q}}\right]$$

$$\hat{a}_p\,\hat{a}^\dagger_{\vec{q}} - \delta^3(\vec{p}-\vec{q})$$

$$= \int d^3q \frac{\vec{q}^2}{2M}\left[\cancel{\hat{a}_{\vec{p}}\,\hat{a}^\dagger_{\vec{q}}\,\hat{a}_{\vec{q}}} - \cancel{\hat{a}_{\vec{p}}\,\hat{a}^\dagger_{\vec{q}}}\,\hat{a}_{\vec{q}} + \delta^3(\vec{p}-\vec{q})\,\hat{a}_{\vec{q}}\right]$$

$$= \frac{\vec{p}^2}{2M}\hat{a}_{\vec{p}}(t)$$

$\Longrightarrow$ Solution : $\hat{a}_{\vec{p}}(t) = e^{-i\frac{\vec{p}^2 t}{2M\hbar}}\,\hat{a}_{\vec{p}}$

$$= e^{-i\frac{E_p t}{\hbar}}\,\hat{a}_{\vec{p}}$$

But then

$$\hat{\psi}(\vec{x},t) = \int d^3p\; u_{\vec{p}}(\vec{x})\;\hat{a}_{\vec{p}}(t)$$

$$= \int d^3p\; u_p(\vec{x})\; e^{-iE_pt/\hbar}\;\hat{a}_{\vec{p}}$$

This means that

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2\vec{\nabla}^2}{2M}\right)\hat{\psi}(\vec{x},t) = \int d^3p\;\hat{a}_{\vec{p}}\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2\vec{\nabla}^2}{2M}\right)\left(u_{\vec{p}}(\vec{x})e^{\frac{-iE_pt}{\hbar}}\right)$$

$$= \int d^3p\;\hat{a}_{\vec{p}}\underbrace{\left(E_{\vec{p}} - \frac{\vec{p}^2}{2M}\right)}_{0}u_{\vec{p}}(\vec{x})\,e^{-iE_pt/\hbar}$$

$$= 0$$

$\Longrightarrow$ $\hat{\psi}(\vec{x},t)$ satisfies the Schrödinger equation!

$\hat{\psi}(\vec{x},t)$ called QUANTUM FIELD

A DIFFERENT VIEW

Consider classical field $\psi(\vec{x},t)$ obeying

$$\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \vec{\nabla}^2}{2M} \right) \psi(\vec{x},t) = 0$$

[for the moment $\hbar$ is just a constant, here everything is still classic]

The Eq. of Motion can be derived from the action

$$S = \int d^4x \underbrace{\hat{\psi}^*(x) \left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \vec{\nabla}^2}{2M} \right] \psi(x)}_{\mathcal{L}}$$

We now quantize this action using canonical quantization

Strategy: 1. Compute conjugate momentum

$$\pi(x) = \frac{\delta \mathcal{L}}{\delta \dot{\psi}}$$

2. Promote $\{\psi(x), \pi(x)\}$ to operators

$$\{\psi(x), \pi(x)\} \longrightarrow \{\hat{\psi}(x), \hat{\pi}(x)\}$$

3. Impose comm. relations

$$[\hat{\psi}(x), \hat{\pi}(y)] = i\hbar\, \delta^3(\vec{x}-\vec{y})$$

Conjugate momentum $\rightarrow$ $\pi(x) = i\hbar \psi^*(x)$

Comm. relations $[\hat{\psi}(x), \hat{\pi}(y)] = i\hbar\, \delta^3(\vec{x}-\vec{y})$

$$\downarrow$$

$$[\hat{\psi}(x), \hat{\psi}^\dagger(y)] = \delta^3(\vec{x}-\vec{y})$$

Same result as before!

Message :

Second quantization in position space is the quantum theory of the field $\psi(x)$ quantized according to canonical quantization and described by a Lagrangian

$$\mathcal{L} = \psi^{\dagger}\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2\vec{\nabla}^2}{2M}\right)\psi$$

Thus

Second quantization $\longleftrightarrow$ Quantum field theory

Relativistic Quantum Fields

We have seen that the KG & Dirac eqs. do not make sense as relativistic one-particle wave equations. We can however use them in second quantization as wave equations of quantum fields.

How? 1. Define operators $\hat{a}^{\dagger}_{\vec{p},\sigma}$ via

$$|p,\sigma\rangle = \hat{a}^{\dagger}_{\vec{p},\sigma}|0\rangle$$

2. Define quantum fields $\hat{\phi}(x)$ via

$$\hat{\phi}(\vec{x}) = \int d^3p \; u_{\vec{p}}(\vec{x}) \; \hat{a}_{\vec{p}}$$

satisfies the wave equation

The Spin-0 field (massive)

In this case only $\sigma = 0$ allowed

$$\Rightarrow |p\rangle = \hat{a}^{\dagger}_{\vec{p}}|0\rangle$$

Lorentz transformation:

$$\hat{U}(\Lambda)|p\rangle = \frac{N_p}{N_{\Lambda p}}|\Lambda p\rangle$$

(because there is no σ quantum#)

use $N_p = \sqrt{\frac{(p_0)^0}{(p)^0}}$

$$= \sqrt{\frac{(\Lambda p)^0}{p^0}}\,|\Lambda p\rangle$$

But then from

$$\hat{U}(\Lambda)|p\rangle = \hat{U}(\Lambda)\,\hat{a}^{\dagger}_p|0\rangle$$

$$\overset{!}{=} \left[\hat{U}(\Lambda)\,\hat{a}^{\dagger}_p\,\hat{U}^{-1}(\Lambda)\right]\underbrace{\hat{U}(\Lambda)|0\rangle}_{|0\rangle}$$

take vacuum to be invariant

We obtain

$$\hat{U}(\Lambda)\hat{a}^{\dagger}_{p}\hat{U}^{-1}(\Lambda) = \sqrt{\frac{(\Lambda p)^0}{p^0}}\,\hat{a}^{\dagger}_{\Lambda p}$$

tr. rule for the creation op.

Field operator : take $u_{\vec{p}}(\vec{x})$ satisfying the KG equation

$\Downarrow$

we can take a plane wave

$$u_{\vec{p}}(\vec{x}) = \frac{e^{i\vec{p}\cdot\vec{x}}}{(2\pi)^{3/2}}$$

$$\Rightarrow \hat{\phi}(\vec{x},t) = \int \frac{d^3p}{(2\pi)^{3/2}}\, e^{i\vec{p}\cdot\vec{x}}\,\hat{a}_{\vec{p}}(t)$$

How does $\hat{\phi}$ transform under a Lorentz transformation?

We use $\hat{U}(\Lambda)\,\hat{a}^\dagger_{\vec{p}}\,\hat{U}^{-1}(\Lambda) = \sqrt{\frac{\omega_{\Lambda p}}{\omega_p}}\,\hat{a}^\dagger_{\Lambda p}$

$$\hat{U}(\Lambda)\,\hat{a}_{\vec{p}}\,\hat{U}^{-1}(\Lambda) = \sqrt{\frac{\omega_{\Lambda p}}{\omega_p}}\,\hat{a}_{\Lambda p}$$

Then

$$\hat{U}(\Lambda)\,\hat{\phi}\,\hat{U}^{-1}(\Lambda) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_p}}\,\hat{U}(\Lambda)\,\hat{a}_{\vec{p}}\,\hat{U}^{-1}(\Lambda)$$

$$= \int \frac{d^3p}{(2\pi)^{3/2}} \frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_p}}\sqrt{\frac{\omega_{\Lambda p}}{\omega_p}}\,\hat{a}_{\Lambda p}$$

$= $ call $q = \Lambda p$ and use

$$\frac{d^3p}{\omega_p} = \frac{d^3q}{\omega_q}$$

$$= \int \frac{d^3q}{(2\pi)^{3/2}} \frac{\omega_p}{\omega_q} \frac{e^{i(\overrightarrow{\Lambda^{-1}q})\cdot\vec{x}}}{\sqrt{2\omega_p}} \sqrt{\frac{\omega_q}{\omega_p}}\, \hat{a}_{\vec{q}}$$

$$= \int \frac{d^3q}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_q}} e^{i(\overrightarrow{\Lambda^{-1}q})\cdot\vec{x}}\, \hat{a}_{\vec{q}}$$

$$= \hat{\phi}(\Lambda^{-1}x)$$

$\Rightarrow$ exactly the transformation we would expect from a scalar field!

Time dependence:

Since we know that $\hat{\phi}(\vec{x},t)$ must satisfy the KG equation, we obtain

$$0 = \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right) \hat{\phi}(\vec{x},t)$$

$$= \int \frac{d^3p}{(2\pi)^{3/2}} e^{i\vec{p}\cdot\vec{x}} \underbrace{\left(\ddot{\hat{a}}_{\vec{p}}(t) + (\vec{p}^2 + m^2)\hat{a}_p(t)\right)}_{\text{must be } 0}$$

Solution:

$$\hat{a}_p(t) = \underbrace{e^{-i\omega_p t}\hat{a}_p^{(+)}}_{\text{positive energy}} + \underbrace{e^{i\omega_p t}\hat{a}_p^{(-)}}_{\text{negative energy}}$$

with $\omega_p = \sqrt{\vec{p}^2 + m^2}$

But then the scalar quantum field is

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_p}} \left[\hat{a}^{(+)}_{\vec{p}} e^{-i(\omega_p t - \vec{p}\cdot\vec{x})} + \hat{a}^{(-)}_{\vec{p}} e^{+i(\omega_p t + \vec{p}\cdot\vec{x})} \right]$$

send $\vec{p} \to -\vec{p}$

and define $px = \omega_p t - \vec{p}\cdot\vec{x}$

$$= \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_p}} \left[\hat{a}^{(+)}_{\vec{p}} e^{-ipx} + \hat{a}^{(-)}_{-\vec{p}} e^{ipx} \right]$$

normalization ensures that

$$\left[\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{y},t) \right] = i\delta^3(\vec{x}-\vec{y})$$

Negative energies & antiparticles

Negative energy part of field:

$$\hat{a}^{(-)}_{\vec{p}}\, e^{i(\omega_p t + \vec{p}\cdot\vec{x})}$$

Stuckelberg (1941) & Feynman (1948):

having a particle with $E = -\omega_p$ moving <u>forward</u> in time is equivalent to have a particle with $E = +\omega_p$ moving <u>backward</u> in time

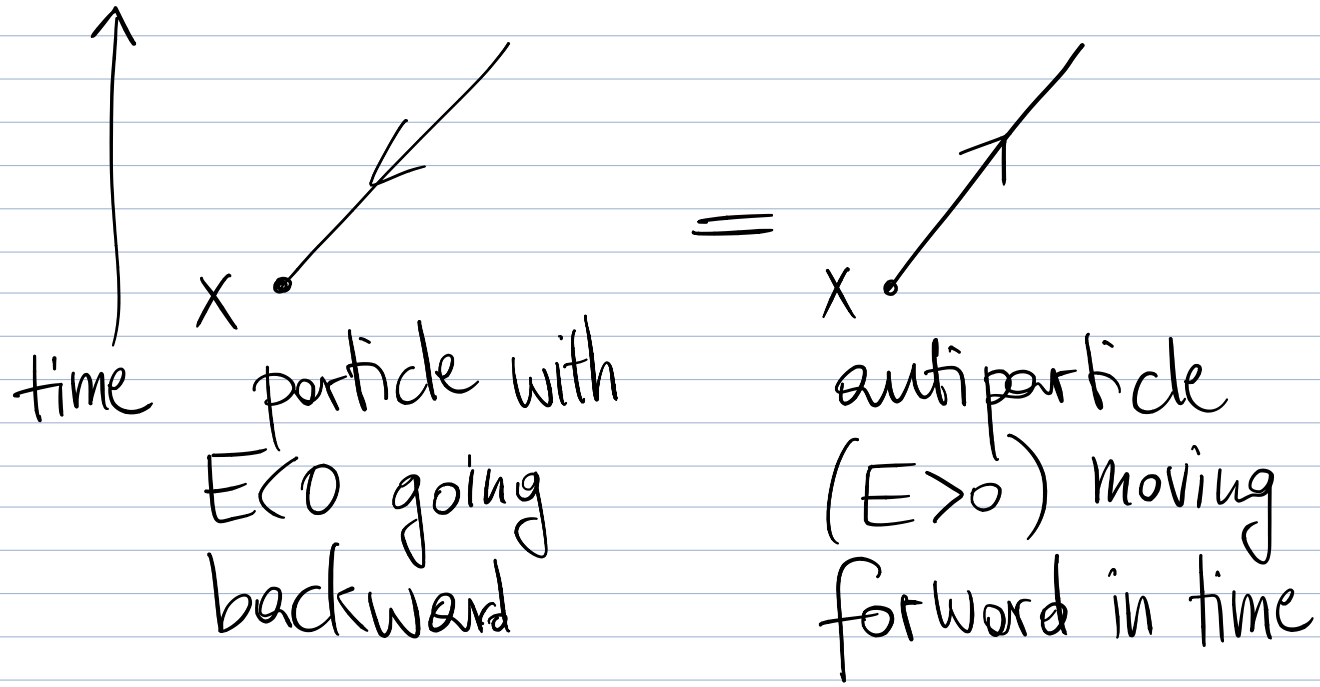
$$e^{i\omega_p t} = e^{-i(-\omega_p)t} = e^{-i(\omega_p)(-t)}$$

negative energy

backward in time

A particle moving backward in time is what we call antiparticle moving forward in time.

It can be shown that if the particle has electric charge q, then the antiparticle has electric charge $-q$.
Automatically the antiparticle must have the same mass as the particle.



Clear that

$$\hat{a}^{(-)}_{-p} = \text{destroys particle with } E<0$$
$$\|$$
$$\text{create antiparticle } (E>0)$$
$$\|$$
$$b^{\dagger}_{p}$$

(we use $b^{\dagger}_{p}$ because it's not a particle bei created, thus I need another operator)

Field theory description

EoM $\to$ KG equation $(\hat{p}^2 - m^2)\phi = 0$

comes from

$$\mathcal{L} = \phi^\dagger (\hat{p}^2 - m^2)\phi = -\phi^\dagger(\Box + m^2)\phi$$

We can now finally explain why we use the $\frac{1}{\sqrt{2\omega_p}}$ normalization factor

$\downarrow$

Look at the Hamiltonian

(equivalent: 00 component energy-momentum tensor)

$$\hat{H} = \int d^3x \left[\hat{\pi}\dot{\hat{\phi}} + \hat{\pi}^\dagger \dot{\hat{\phi}}^\dagger - \mathcal{L}[\hat{\phi}] \right]$$

Explicitly: to compute $\hat{\pi}$, more convenient to rewrite $\mathcal{L}$ as

$$\mathcal{L} = -\phi^\dagger(\Box + m^2)\phi \overset{\text{by parts}}{=} \partial_\mu \phi^\dagger \partial_\mu \phi - m^2|\phi|^2$$

$$= |\dot{\phi}|^2 - |\vec{\nabla}\phi|^2 - m^2|\phi|^2$$

$$\Downarrow$$

$$\pi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}} = \dot{\phi}^\dagger \;; \quad \pi^\dagger = \frac{\delta \mathcal{L}}{\delta \dot{\phi}^\dagger} = \dot{\phi}$$

Putting all together:

$$H = \int d^3x \left[2|\dot{\phi}|^2 - |\dot{\phi}|^2 + |\vec{\nabla}\phi|^2 + m^2|\phi|^2 \right]$$

$$= \int d^3x \left[|\dot{\phi}|^2 + |\vec{\nabla}\phi|^2 + m^2|\phi|^2 \right]$$

Now write

$$\phi = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_p}} \left[\underbrace{\hat{a}_p e^{-i\omega_p t}}_{=\hat{a}_p(t)} e^{i\vec{p}\cdot\vec{x}} + \underbrace{b_p^\dagger e^{i\omega_p t}}_{=\hat{b}_p^\dagger(t)} e^{-i\vec{p}\cdot\vec{x}} \right]$$

change $\vec{p} \to -\vec{p}$

$$= \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_p}} \left[\hat{a}_p(t) + \hat{b}_{-p}^\dagger(t) \right] e^{i\vec{p}\cdot\vec{x}}$$

Using this, easy to compute that

$$\hat{H} = \int d^3p \left\{ \underbrace{\frac{\omega_p}{2}\left[\hat{a}_p^\dagger \hat{a}_p + \hat{b}_{-p}\hat{b}_{-p}^\dagger - \hat{a}_p^\dagger \hat{b}_p - \hat{b}_{-p}^\dagger \hat{a}_p\right]}_{\text{from } |\dot{\hat{\phi}}|^2} \right.$$

$$\left. + \underbrace{\frac{\overset{\omega_p^2}{\vec{p}^2 + m^2}}{2\omega_p}\left[\hat{a}_p^\dagger \hat{a}_p + \hat{b}_{-p}\hat{b}_{-p}^\dagger + \hat{a}_p^\dagger \hat{b}_{-p} + \hat{b}_{-p}^\dagger \hat{a}_p\right]}_{\text{from } |\vec{\nabla}\hat{\phi}|^2 + m^2|\hat{\phi}|^2} \right\}$$

$$= \int d^3p \; \omega_p \left(\hat{a}_p^\dagger \hat{a}_p + \hat{b}_{-p}\hat{b}_{-p}^\dagger\right)$$

change again $\vec{p} \to -\vec{p}$

$$= \int d^3p \; \omega_p \left(\hat{a}_p^\dagger \hat{a}_p + \hat{b}_p \hat{b}_p^\dagger\right)$$

use $\left[\hat{b}_p, \hat{b}_q^\dagger\right] = \delta^3(\vec{p}-\vec{q})$

$$= \int d^3p\ \omega_p \left\{ \hat{a}_p^\dagger \hat{a}_p + \hat{b}_p^\dagger \hat{b}_p + \delta^3(0) \right\}$$

$\hat{N}_p$ for particles

$\hat{N}_p$ for antiparticles

Vacuum energy

$\Rightarrow$ the factor $\frac{1}{\sqrt{2\omega_p}}$ in the field allows for the correct energy normalization!

Vacuum energy $\rightarrow$ infinite contribution

The spin-1/2 field (massive)

We procede analogously to the spin-0 field, but now there is a non-trivial Lorentz transformation:

$$\hat{U}(\Lambda)\,|p,\sigma\rangle = \sqrt{\frac{\omega_{\Lambda p}}{\omega_p}} \sum_{\sigma'} D_{\sigma\sigma'}\,|\Lambda p,\sigma'\rangle$$

Rotation metrices in 2-dim ← spinor representation of SO(3)

Quantum field: almost as before

Logic: must be a simultaneous solution of the KG & Dirac equations

$$\hat{\psi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_p}} \sum_\sigma \left[u_p^\sigma \hat{a}_p^\sigma e^{-ipx} + v_p^\sigma \hat{b}_p^{\sigma\dagger} e^{ipx} \right]$$

Sum over spins $\sigma = \pm$

must satisfy the Dirac eq.

$\Rightarrow$ u_p, v_p are the positive/negative energy spinors we have already computed

The photon field

What is the wave eq. for a spin-1 field?

We can procede as usual: we know that $S=1$ is contained in the Lorentz representations

$$(1,0) \rightarrow \phi^{(+)}_{\mu\nu} \quad \text{antisymmetric}$$

$$\left(\tfrac{1}{2},\tfrac{1}{2}\right) \rightarrow \phi_{\mu}$$

$$(0,1) \rightarrow \phi^{(-)}_{\mu\nu} \quad \text{antisymmetric}$$

Again, we apply $\hat{p}_\mu$ in all possible ways

$$\begin{cases} \left.\begin{aligned} \hat{p}^\mu \phi^{(+)}_{\mu\nu} &= a\,\phi_\nu \\ \hat{p}^\mu \phi^{(-)}_{\mu\nu} &= a\,\phi_\nu \end{aligned}\right\} \\ \hat{p}_\mu \phi_\nu - \hat{p}_\nu \phi_\mu = a\,\phi_{\mu\nu} \end{cases}$$

clear that only $\phi^{(+)}_{\mu\nu} + \phi^{(-)}_{\mu\nu} \equiv \phi_{\mu\nu}$ can enter

Then $\underbrace{\hat{p}^\mu \hat{p}^\nu}_{\text{symm}} \underbrace{\phi_{\mu\nu}}_{\text{antisym.}} = 0 = a\,\hat{p}^\mu \phi_\mu$

$$\Downarrow$$

$$\partial^\mu \phi_\mu = 0$$

and

$$\hat{p}^\mu \phi_{\mu\nu} = \frac{\hat{p}^\mu}{a}\left(\hat{p}_\mu \phi_\nu - \hat{p}_\nu \phi_\mu\right) = a\,\phi_\nu$$

$$\Longrightarrow \quad \hat{p}^2 \phi_\nu - \hat{p}_\nu \underbrace{\hat{p}^\mu \phi_\mu}_{0} = a^2 \phi_\nu$$

$$\Rightarrow \left(\hat{p}^2 - a^2\right) \phi_\nu = 0$$

this is KG if we take $a^2 \equiv m^2$

To make contact with usual notation, we redefine

$$\phi_{\mu\nu} \rightarrow \frac{i}{a} \phi_{\mu\nu}$$

in such a way that

$$\begin{cases} \phi_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu \\ -\partial^\mu \phi_{\mu\nu} = m^2 \phi_\nu \end{cases}$$

wave eqs. for $S=1$ particle

In the $m \rightarrow 0$ limit

$\partial^\mu \phi_{\mu\nu} = 0 \rightarrow$ <u>Maxwell eq. in vacuum!</u>

What about the photon quantum field?

We would like

$$\hat{A}_\mu = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_p}} \sum_{h=\pm} \Big[\epsilon^h_\mu(\vec{p})\, \hat{a}_{\vec{p}}\, e^{-ipx} + \epsilon^{h*}_\mu(\vec{p})\, \hat{a}^\dagger_{\vec{p}}\, e^{ipx} \Big]$$

with $\epsilon^i_\mu(\vec{p})$ $\begin{cases} \text{4-vectors} \\ \text{Satisfying Maxwell eq.} \end{cases}$

1. Satisfying Maxwell eqs. is easy:

$$p^\mu \epsilon^\pm_\mu(\vec{p}) = 0$$

2. Having a 4-vector is impossible

$\downarrow$

Why?

Remember that we have found that the polarization vectors transform under a Lorentz tr. as

$$\Lambda \epsilon^{\pm}(\vec{p}) = e^{\pm i\theta}\left(\epsilon^{\pm}(\Lambda p) + f_{\pm}\Lambda p\right)$$

rotation of little group

$$\Rightarrow \epsilon^{h}(\vec{p}) = e^{ih\theta}\left[\Lambda^{-1}\epsilon^{h}(\Lambda p) + f_h\, p\right]$$

What about the creation/annihilation ops?

Remember that

$$\hat{\mathcal{U}}(\Lambda)|p,h\rangle = e^{ih\theta}\sqrt{\frac{\omega_{\Lambda p}}{\omega_p}}\,|\Lambda p, h\rangle$$

$h =$ helicity

in such a way that

$$\hat{U}(\Lambda)\, \hat{a}^{\dagger}_{p,h}\, \hat{U}^{-1}(\Lambda) = e^{ih\theta}\, \hat{a}^{\dagger}_{\Lambda p,h} \sqrt{\frac{\omega_{\Lambda p}}{\omega_p}}$$

and

$$\hat{U}(\Lambda)\, \hat{a}_{p,h}\, \hat{U}^{-1}(\Lambda) = e^{-ih\theta}\, \hat{a}_{\Lambda p,h} \sqrt{\frac{\omega_{\Lambda p}}{\omega_p}}$$

Putting all together:

$$\hat{U}(\Lambda)\hat{A}_\mu \hat{U}^{-1}(\Lambda) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_p}} \sum_{h=\pm} \sqrt{\frac{\omega_{\Lambda p}}{\omega_p}}$$

$$\left\{ \epsilon_\mu^h(\vec{p})\, \hat{a}_{\Lambda p,h}\, e^{-ih\theta}\, e^{-ipx} + \epsilon_\mu^{h*}(\vec{p})\, \hat{a}^\dagger_{\Lambda p,h}\, e^{ih\theta}\, e^{ipx} \right\}$$

use

$$e^{-ih\theta}\epsilon_\mu^h(\vec{p}) = (\Lambda^{-1})_\mu{}^\nu \epsilon_\nu^h(\Lambda p) + f_h p_\mu$$

⊕ call $q = \Lambda p$

⊕ use $\frac{d^3p}{\omega_p} = \frac{d^3q}{\omega_q}$ (Lorentz invariant)

We get

$$\hat{U}(\Lambda)\hat{A}_\mu\hat{U}^{-1}(\Lambda) = (\Lambda^{-1})_\mu{}^\nu \hat{A}_\nu(\Lambda^{-1}x) + \partial_\mu \lambda_\Lambda$$

horrible function that depends on $f_\pm$ etc.

<u>But this is a gauge transformation!</u>

Conclusions

1. Impossible to describe a photon (2 dof) with a 4-vector, the Lorentz tr. involves also a gauge transformation

2. Maxwell eq. $\partial^\mu F_{\mu\nu} = 0$ is the only wave eq. invariant under gauge transformation

$\Rightarrow$ there is really no other possibility that obtain EM for a massless photon

3. The same reasoning applies for a spin-2 particle (= graviton).
The resulting theory is <u>General Relativity</u>